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LECTURE NOTES

for

MULTIWAVELETS AND THEIR APPLICATIONS

by

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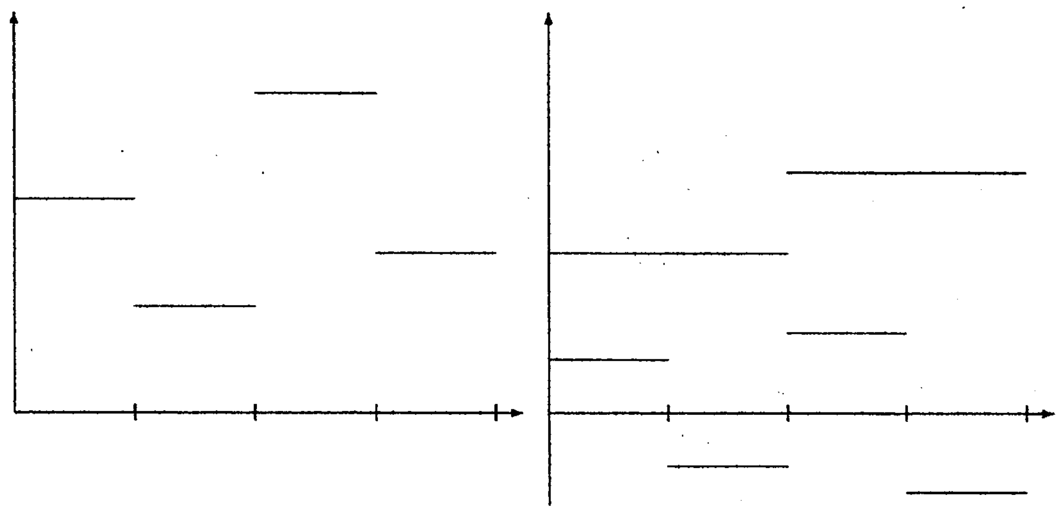
1 Introduction to Scaling Vectors and Multiwavelets



$$
\begin{aligned}
f_2 &= f_1 + g_2 \\
V_2 &= V_1 \oplus W_2
\end{aligned}
$$

Haar Basis: $\phi^H(x) =$ $\qquad\qquad$ $\psi^H(x) =$

$$
\begin{aligned}
f_2(x) &= a\,\phi^H(4x) + b\,\phi^H(4x-1) + c\,\phi^H(4x-2) + d\,\phi^H(4x-3) \\
&= \left(\tfrac{a+b}{2}\right)\phi^H(2x) + \left(\tfrac{c+d}{2}\right)\phi^H(2x-1) \\
&\quad + \left[a - \tfrac{a+b}{2}\right]\psi^H(2x) + \left[d - \tfrac{c+d}{2}\right]\psi^H(2x-1)
\end{aligned}
$$

$$\begin{aligned} \phi^H(x) &= \phi^H(2x) + \phi^H(2x-1) \\ \psi^H(x) &= \phi^H(2x) - \phi^H(2x-1). \end{aligned}$$

Two-Scale Dilation Equations/Refinement Equations

$$\boxed{\begin{array}{ll} \phi^H\text{:} & \textit{Haar scaling function} \\ \psi^H\text{:} & \textit{Haar wavelet} \end{array}}$$

If $f(x) \in L^2(\mathbb{R})$, i.e., $\|f\|_2 := \sqrt{\int_{-\infty}^{+\infty} |f(x)|^2\,dx} < \infty$, then

$$\boxed{f(x) = \sum_{k=-\infty}^{+\infty} \sum_{\ell=-\infty}^{+\infty} c_{k\ell}\psi^H(2^k x - \ell)},$$

where $\mathbf{c} := (c_{k\ell}) \in \ell^2(\mathbb{R})$, that is, $\sum_{k=-\infty}^{+\infty} \sum_{\ell=-\infty}^{+\infty} |c_{k\ell}|^2 < \infty$.

$$\boxed{L^2(\mathbb{R}) = \bigoplus_{k=-\infty}^{+\infty} W_k},$$

with

$$\boxed{W_k = \mathrm{cl}_{L^2(\mathbb{R})}\ \mathrm{span}\ \{\psi^H_{k\ell} = 2^{k/2}\psi^H(2^k \cdot - \ell) \,|\, k, \ell \in \mathbb{Z}\}}.$$

Wavelet Spaces

$$\boxed{V_k = \mathrm{cl}_{L^2(\mathbb{R})}\ \mathrm{span}\ \{\phi^H_{k\ell} = 2^{k/2}\phi^H(2^k \cdot - \ell) \,|\, k, \ell \in \mathbb{Z}\}}.$$

Two-Scale Principal Shift-Invariant Spaces with generator ϕ^H

Note that

$$\boxed{\begin{array}{ccccc} V_k & = & V_{k-1} & \oplus & W_k, \quad k \in \mathbb{Z} \\ \text{fine scale} & \rightarrow & \text{coarse scale} & \oplus & \text{detail correction} \end{array}}$$

Properties of the Haar Scaling Function and Wavelet

- $\int_{-\infty}^{+\infty} \phi^H(x)dx = 1$
- *Vanishing Moment*: $\int_{-\infty}^{+\infty} \psi^H(x)dx = 0$
- *Two-Scale Dilation Equations*:

$$\boxed{\begin{array}{rcl} \phi^H(x) & = & \sum_\ell g_\ell \phi^H(2x-\ell) \\ \psi^H(x) & = & \sum_\ell h_\ell \phi^H(2x-\ell) \end{array}}$$

(Here $g_0 = g_1 = h_0 = -h_1 = 1$.)

- *Compact Support*: both ϕ^H and ψ^H vanish outside a closed interval of finite length;
- *Smoothness*: ϕ^H and ψ^H piecewise continuous;
- *Orthogonality*:

$$\boxed{\begin{array}{rcl} \int_{-\infty}^{+\infty} \psi^H(2^k x-\ell)\psi^H(2^m x-n)dx & = & \begin{cases} 2^k & k=m \text{ and } \ell=n \\ 0 & \text{otherwise;} \end{cases} \\ \int_{-\infty}^{+\infty} \phi^H(2^k x-\ell)\phi^H(2^k x-n)dx & = & \begin{cases} 2^k & \ell=n \\ 0 & \text{otherwise;} \end{cases} \\ \int_{-\infty}^{+\infty} \phi^H(2^k x-\ell)\psi^H(2^k x-n)dx & = & 0; \end{array}}$$

- The one-parameter family $\{\phi^H(\cdot-\ell) \mid \ell \in \mathbb{Z}\}$ forms an orthogonal basis of V_0. (Thus, the one-parameter family $\{\phi^H(2^k\cdot-\ell) \mid \ell \in \mathbb{Z}\}$ an orthogonal basis of V_k, for $k \in \mathbb{Z}$.)
- The two-parameter family $\{\psi^H_{k\ell} := 2^{k/2}\psi(2^k\cdot-\ell) \mid k,\ell \in \mathbb{Z}\}$ forms an orthonormal basis of $L^2(\mathbb{R})$.

Example 1.1 (Hat-function) *Let*

$$h(x) := \begin{cases} 1 - |x|, & |x| \leq 1 \\ 0, & \textit{otherwise.} \end{cases}$$

The hat function is a linear B-spline with knots $\{(-1,0),(0,1),(1,0)\})$

Define the principal shift-invariant space $V_0[h]$ with generator h by

$$V_0[h] := \mathrm{cl}_{L^2(\mathbb{R})} \operatorname{span} \{h(\cdot - \ell) \mid \ell \in \mathbb{Z}\}$$

Exercise : *Show that h generates an MRA!*

Short-hand notation: $\tau[f] := \operatorname{span} \{f(\cdot - \ell) \mid \ell \in \mathbb{Z}\}$ and $\sigma[f] := \mathrm{cl}_{L^2(\mathbb{R})}\tau[f]$.

Note that

$$h(x) = \frac{1}{2}h(2x-1) + h(2x) + \frac{1}{2}h(2x+1) \quad \text{(Verify this!)} \tag{1.3}$$

but

$$\int_{-1}^{1} h(x)h(x \pm 1)dx = 1/6 \neq 0.$$

Hence, the integer shifts are *not* orthogonal to each other. In other words, the MRA is not *orthogonal.*

Does there exist a wavelet ψ with smallest possible support, say [-1/2,1]? If so, then

$$\psi(x) = a \cdot h(2x) + b \cdot h(2x-1), \qquad a, b \in \mathbb{R}.$$

Now, $\int_{-1}^{1} h(x) \cdot \psi(x)dx = 0$; thus, a possible choice for a and b is $a = -(3/5)b$. (Verify this!) Choosing $a = 1$ gives

$$\psi(x) = h(2x) - (5/3) \cdot h(2x-1).$$

However, ψ is neither orthogonal to its shifts nor to the nonzero shifts of h. (Show this!)

Exercise : *Does there exist a wavelet ψ associated with h and supported on $[-1,1]$ that is orthogonal to its nonzero shifts as well as the shifts of h?*

Example 1.2 *Cardinal B-splines.*

Terminology:

- The support of a function f, supp$f$, is the largest closed set for which $f(x) \neq 0$.
- P_n: vector space of all real polynomials of degree at most n;
- C^n: vector space of all m-times continuously differentiable functions; also, let $C = C^0$.
- C^∞: vector space of all infinitely differentiable functions.

Definition 1.4 (Cardinal B-Splines) *Let* $N_1(x) := \chi_{[0,1)}$. *For* $m \geq 2$, *define*

$$\boxed{\begin{aligned} N_m(x) := (N_{m-1} * N_1)(x) &= \int_{-\infty}^{\infty} N_{m-1}(x-t)N_1(t)dt \\ &= \int_0^1 N_{m-1}(x-t)dt. \end{aligned}}$$

Exercise : *Calculate N_2 and N_3!*

Exercise : *Show that* supp$N_m = [0, m)$! *(Use induction on m!)*

Exercise : *Verify, at least formally, that* $\lim_{m\to\infty} N_m(x) \in C^\infty$!

Let $V_0^m := \sigma[N_m]$, for a fixed $m \geq 1$. Then V_0^m consists of all functions $f \in C^{m-2} \cap L^2(\mathbb{R})$ whose restriction to an interval of the form $[\ell, \ell+1)$, $\ell \in \mathbb{Z}$, is a polynomial of degree at most $m-1$ (Verify this!). Thus, for all $k \in \mathbb{Z}$:

$$\boxed{V_k^m = \{f \in C^{m-2} \cap L^2(\mathbb{R}) \mid f|_{[\ell,\ell+1)} \in P_{m-1}, \ell \in \mathbb{Z}\}.} \tag{1.4}$$

Employing properties of the cardinal B-splines one shows that the collection $\{N_m(\cdot - \ell) \mid \ell \in \mathbb{Z}\}$ forms a Riesz basis of V_0^m. (cf. [7])

Example 1.3 *The Hat-function revisited. (Donovan–Geronimo–Hardin–Roach)*

Definition 1.5 (Finitely Generated Shift-Invariant Space) *A space V is called a finitely generated shift-invariant space if there is a finite set $\phi = (\phi^1, \ldots, \phi^r)^T$ of L^2-functions such that*

$$V = \mathrm{cl}_{L^2(\mathbb{R})} \operatorname{span}\{\phi^i(\cdot - \ell) \mid i = 1, \ldots, r; \ell \in \mathbb{Z}\}. \tag{1.5}$$

We write

$$\tau[\phi] := \operatorname{span}\{\phi^i(\cdot - \ell) \mid i = 1, \ldots, r; \ell \in \mathbb{Z}\} \tag{1.6}$$

and

$$\sigma[\phi] := \mathrm{cl}_{L^2(\mathbb{R})}\, \tau[\phi]. \tag{1.7}$$

Let $Df := f(\cdot/2)$. The space V is called refinable *if*

$$\boxed{D(V) \subset V.} \tag{1.8}$$

Remark: If $V = \sigma[\phi]$ then V is refinable iff

$$\boxed{\phi(x) = \textstyle\sum_\ell c_\ell \phi(2x - \ell), \quad c_\ell \in \mathbb{R}^{r \times r}.} \tag{1.9}$$

Let $w \in L^2(\mathbb{R})$ be supported on $[0,1]$ and let V be the space generated by h and w:

$$V := \sigma[h, w]$$

Introduce new generators by

$$\begin{aligned} \phi^1 &:= w \\ \phi^2 &:= (I - P_{\sigma[w]})h \\ &= h - \frac{\langle h, w\rangle}{\langle w, w\rangle} w - \frac{\langle h, w(\cdot + 1)\rangle}{\langle w, w\rangle} w(\cdot + 1) \end{aligned}$$

Need

$$\langle \phi^2, \phi^2(\cdot - 1)\rangle = \langle h, h(\cdot - 1)\rangle - \frac{\langle h, w\rangle \langle w, h(\cdot - 1)\rangle}{\langle w, w\rangle} = 0$$

$\phi = (\phi^1, \phi^2)^T$ will be orthogonal iff

$$\boxed{\langle h, h(\cdot - 1)\rangle = \tfrac{\langle h, w\rangle\langle w, h(\cdot - 1)\rangle}{\langle w, w\rangle}.} \tag{1.10}$$

Let $q(x) := \begin{cases} x(1-x) & 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases} = x(1-x)\chi_{[0,1]} = x(1-x)^+.$

Choose $w(x) := q(x) + \alpha q^2(x)$. Then

$$\begin{aligned} \langle h, w \rangle &= \langle w, h(\cdot - 1)\rangle \\ &= (1/60)(5+\alpha) \quad \text{(Verify this!)} \\ \langle w, w \rangle &= (1/630)(21 + 9\alpha + \alpha^2) \quad \text{(Verify this!)} \end{aligned}$$

Hence, Eqn. (1.10) reads:

$$\alpha^2 + 30\alpha + 105 = 0,$$

or $\alpha = -15 \pm 2\sqrt{30}$. Choose $(-)$-sign!

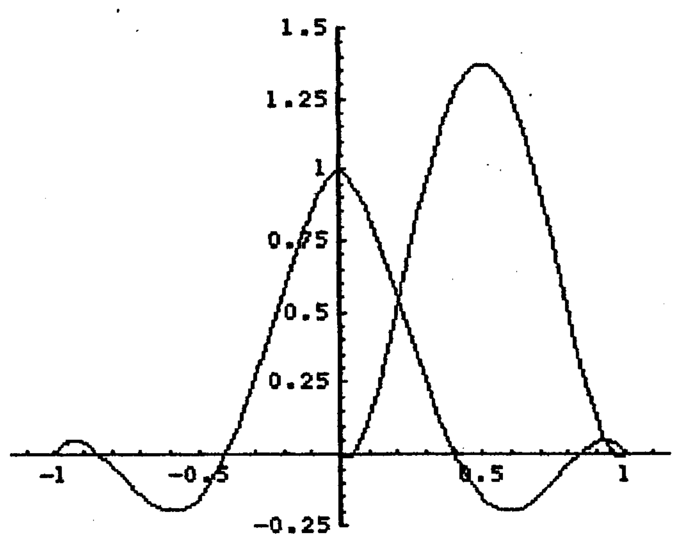


Figure 1: The orthogonal generators ϕ^1 and ϕ^2.

Exercise : *Repeat the above example with* $w(x) := 4x(1-x)\chi_{[0,1]}$*!*

Definition 1.6 (Multiresolution Analysis of Multiplicity r) *A multiresolution analysis of multiplicity r of $L^2(\mathbb{R})$ consists of a sequence of approximation spaces $\{V_k\}_{k\in\mathbb{Z}}$ with the properties:*

Nestedness: *For all $k \in \mathbb{Z}$, $V_k \subseteq L^2(\mathbb{R})$ and $V_k \subseteq V_{k+1}$.*

Approximation Property: *$\bigcup_{k\in\mathbb{Z}} V_k$ is dense in $L^2(\mathbb{R})$ and $\bigcap_{k\in\mathbb{Z}} V_k = \{0\}$.*

Shift-invariance: *If $f \in V_k$ then $f(\cdot - \ell) \in V_k$, for all $\ell \in \mathbb{Z}$.*

Dilation-invariance: *If $f \in V_k$ then $f(2\cdot) \in V_{k+1}$.*

Basis: *There exists a* finite *collection of L^2-functions $\phi := \{\phi^i : i = 1, \ldots, r\}$, called a* scaling vector *or* multiscaling function *with the property that*

1.

$$\boxed{V_0 = V_0[\phi] := \mathrm{cl}_{L^2(\mathbb{R})} \operatorname{span} \{\phi(\cdot - \ell) \mid \ell \in \mathbb{Z}\}}. \tag{1.11}$$

Finitely Generated Shift-invariant Space with generator ϕ

2. The scaling vector ϕ is required to have stable *integer shifts:*

$$R_1 \sum_{\ell\in\mathbb{Z}} \|C_\ell\|^2 \leq \|\sum_{\ell\in\mathbb{Z}} C_\ell \phi(\cdot - \ell)\|_2^2 \leq \sum_{\ell\in\mathbb{Z}} \|C_\ell\|^2, \tag{1.12}$$

for positive constants R_1 and R_2 and all square-summable $r \times r$ matrices $(C_\ell)_{\ell\in\mathbb{Z}}$.

Here

$$\boxed{\|\phi\|_2 := \sqrt{\textstyle\int_{\mathbb{R}} \phi(x)\, \phi^T(x)\, dx}.} \tag{1.13}$$

(The transpose of a vector or matrix is denoted by ^{T}.)

An MRA is called orthogonal *iff the integer shifts of the scaling vector ϕ form an* orthogonal *basis of V_0.*

Definition 1.7 (Multiwavelet) *A* finite *collection of L^2-functions $\psi := \{\psi^i : i = 1, \ldots, r\}$ is called a* multiwavelet *if the two-parameter family $\{\psi_{k\ell} := 2^{k/2}\psi(2^k \cdot - \ell) : k, \ell \in \mathbb{Z}\}$ forms an orthonormal, or more generally, an unconditional basis of $L^2(\mathbb{R})$.*

The nestedness of the spaces V_k implies that ϕ satisfies a *two-scale matrix dilation equation* or *matrix refinement equation*

$$\boxed{\phi(x) = \sum_{\ell\in\mathbb{Z}} G_\ell\, \phi(2x-\ell).} \tag{1.14}$$

$\{C_\ell\}_{\ell\in\mathbb{Z}}$ $r\times r$ matrices, satisfying $\sum_{\ell\in\mathbb{Z}} \|C_\ell\|^2 < \infty$.

Denote the L^2-orthogonal complement of V_k in V_{k+1} by W_{k+1}. The existence of a multiwavelet is paramount to finding $r\times r$ matrices $(H_\ell)_{\ell\in\mathbb{Z}}$ with the properties that

$$\boxed{\sum_{\ell\in\mathbb{Z}} G_\ell\, G^T_{\ell-2\ell'} = \delta_\ell\, I_{r\times r},} \tag{1.15}$$

(orthogonality of integer shifts for ϕ)

$$\boxed{\sum_{\ell\in\mathbb{Z}} H_\ell\, H^T_{\ell-2\ell'} = \delta_\ell\, I_{r\times r},} \tag{1.16}$$

(orthogonality of integer shifts for ψ)

$$\boxed{\sum_{\ell\in\mathbb{Z}} G_\ell\, H^T_{\ell-2\ell'} = O_{r\times r},} \tag{1.17}$$

(orthogonality between ϕ and ψ)

$$\boxed{\sum_{\ell\in\mathbb{Z}} G^T_{m-2\ell}\, G_{n-2\ell'} + H^T_{m-2\ell}\, H_{n-2\ell'} = \delta_{mn} I_{r\times r}, \quad m,n\in\mathbb{Z},} \tag{1.18}$$

($V_0 + W_1 = V_1$)

Here $I_{r\times r}$ and $O_{r\times r}$ denotes the $r\times r$ identity, respectively, zero matrix.

Exercise : *Show that the above equations for the matrices $(H_\ell)_{\ell\in\mathbb{Z}}$ follow from the conditions given below.*

- $\boxed{\psi(x) = \sum_{\ell\in\mathbb{Z}} H_\ell \phi(2x-\ell);}$
- $\boxed{\langle\phi(2^k\,\cdot - \ell), \phi(2^k\,\cdot - \ell')\rangle = 2^k \delta_{\ell\ell'} I_{r\times r},}$
- $\boxed{\langle\psi(2^k\,\cdot - \ell), \psi(2^{k'}\,\cdot - \ell')\rangle = 2^k \delta_{kk',\ell\ell'} I_{r\times r},}$
- $\boxed{\langle\phi(2^k\,\cdot - \ell), \psi(2^{k'}\,\cdot - \ell')\rangle = O_{r\times r}.}$

(Here we defined $\langle\varphi_1,\varphi_2\rangle := \int_{\mathbb{R}} \varphi_1(x)\,\varphi_2^T(x)\,dx$.)

2 Construction of Daubechies and Spline Wavelets

Objective: Construct a *family* of L^2-functions ϕ and ψ with the properties:

1. ϕ generates an ***orthogonal*** MRA with ψ being the associated wavelet. This means that $\phi_{k\ell}$ and ψ_{mn} satisfy the orthogonality relations.

2. The two-parameter family $\{\psi_{k\ell} := 2^{k/2}\psi(2^k \cdot -\ell) \mid k, \ell \in \mathbb{Z}\}$ forms an orthonormal basis of $L^2(\mathbb{R})$.

3. $\int_{\mathbb{R}} \phi(x)dx \neq 0$. (Necessary for technical reasons.)

4. ϕ and ψ satisfiy two-scale dilation equations of the form

$$\begin{aligned} \phi(x) &= \textstyle\sum_\ell g_\ell \phi(2x-\ell) \\ \psi(x) &= \textstyle\sum_\ell h_\ell \phi(2x-\ell) \end{aligned}$$

5. ϕ and ψ have compact support. (This implies that the sums in the above dilation equations are *finite*.)

6. Vanishing moments for ψ: $\int_{\mathbb{R}} x^p \psi(x)dx = 0$, for $p = 0, 1, \ldots, N-1$, $N \geq 1$.[3] Geometrically speaking this means that ψ is orthogonal to the space P_{N-1} of real polynomials of degree at most $N-1$:

$$W_k \perp P_N. \qquad k \in \mathbb{Z}.$$

7. ϕ and thus ψ should have some degree of differentiability which implies that ψ will have a certain number of vanishing moments.

[3] Vanishing moments are related to the regularity, i.e., the degree of differentiability, of the function ψ, and thus also ϕ. The number of vanishing moments is also connected with the approximation order: If ψ has N vanishing moments, then ϕ (!) reproduces polynomials up to degree $N-1$, that is,

$$x^p = \sum_{finite} a_\ell \phi(x-\ell), \qquad p = 0, 1, \ldots, N-1.$$

Short excursion into Fourier Theory:

Let $f \in L^2(\mathbb{R})$. The *Fourier transform* $\mathcal{F}(f)$ *of* f is defined by

$$\boxed{\mathcal{F}(f)(\omega) := \int_{\mathbb{R}} e^{i\omega x} f(x)dx.} \tag{2.1}$$

Remarks:

- Some authors put 2π into the argument of the exponential or normalize the integral by $1/(2\pi)$ or $1/\sqrt{2\pi}$.
- For short we write $\hat{f}$ instead of $\mathcal{F}(f)$.

Facts from Fourier Theory

- $\hat{f} \in L^2(\mathbb{R})$
- *Parsival's Identity:* $\langle f, g\rangle = 1/(2\pi)\,\langle \hat{f}, \hat{g}\rangle$, where $\langle f, g\rangle := \int_{\mathbb{R}} f(x)g(x)dx$. In particular, $\|f\|_2 = 1/\sqrt{2\pi}\|\hat{f}\|_2$.
- The Fourier transform is a one-to-one mapping of $L^2(\mathbb{R})$ onto itself whose inverse is given by

$$\boxed{\check{f}(x) := \mathcal{F}^{-1}(f)(x) = \tfrac{1}{2\pi}\int_{\mathbb{R}} e^{-i\omega x} \hat{f}(\omega)d\omega.} \tag{2.2}$$

- If f is compactly supported then $\hat{f}$ is not, and vice-versa. (*Uncertainty Principle of Fourier Analysis*: f cannot be band- limited in frequency and time.)
- *Poisson Summation Formula:*

$$\boxed{\sum_{\ell=-\infty}^{+\infty} f(x+2\pi\ell) = \tfrac{1}{2\pi}\sum_{\ell=-\infty}^{+\infty} \hat{f}(\ell)e^{i\ell x}.} \tag{2.3}$$

  (Provided both sums converge).

- $\boxed{|\hat{f}(\omega)| \le C(1+|\omega|)^{-m} \text{ then } f \in C^{m-1}.}$

One way to construct the family of Daubechies scaling functions and wavelets is via Fourier analysis.

Construct a function ϕ that generates an orthogonal MRA on $L^2(\mathbb{R})$. Note that

$$\phi(x) = \sum_{\ell} g_\ell \phi(2x - \ell) \tag{2.4}$$

defines an operator $T : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ by

$$\boxed{(Tf_0)(x) := \textstyle\sum_\ell g_\ell f_0(2x - \ell),} \tag{2.5}$$

provided that $\sum_\ell |g_\ell|^2 < \infty$. Setting $f_1 := Tf$, and applies T to f_1 one obtains a sequence $\{f_n\}_{n\geq 0}$. **Idea:** Define ϕ as the limit – if it exists – of this sequence:

$$\boxed{\|\phi - f_n\|_2 \to 0 \quad \text{as } n \to \infty.} \tag{2.6}$$

(This ϕ is then the *fixed point* of T: $\phi = T\phi$.)

Take the Fourier transform of Eqn (2.4):

$$\boxed{\hat{\phi}(\omega) = \left[\tfrac{1}{2} \textstyle\sum_\ell g_\ell e^{i\ell\omega}\right] \hat{\phi}(\omega/2).} \tag{2.7}$$

The expression

$$\boxed{m_0(\omega) := \tfrac{1}{2} \textstyle\sum_\ell g_\ell e^{i\ell\omega}} \tag{2.8}$$

is called the *two-scale symbol of* ϕ.

Iterating Eqn (2.7) one obtains, at least formally

$$\boxed{\hat{\phi} = \textstyle\prod_{n=1}^{\infty} m_0(\omega/2^n)\hat{\phi}(0).} \tag{2.9}$$

Suppose, w.l.o.g., that $\hat{\phi}(0) = \int_{\mathbb{R}} \phi(x)dx = 1$. *If* the infinite product in Eqn. (2.9) converges pointwise for all $\omega \in \mathbb{R}$ (this means that one has to impose **growth conditions** on the coefficients $\{c_\ell\}$) to the Fourier transform of a *continuous function*, then we may define ϕ by

$$\phi(x) = \mathcal{F}^{-1}\left(\prod_{n=1}^{\infty} m_0(\omega/2^n)\right). \tag{2.10}$$

Growth condition: There exists an $\eta > 0$ such that $\sum_\ell |g_\ell| \, |\ell|^\eta < \infty$.

Existence of Function: Suppose that

- $\sum_\ell g_{\ell-2i} g_{\ell-2j} = \delta_{ij}$; (Orthogonality to translates!)
- $\sum_\ell g_\ell = 2$; ($\hat{\phi}(0) = 1$!)

and that

$$\boxed{m_0(\omega) = \left[(1+e^{i\omega})/2\right]^N \; M(\omega), \quad N \geq 1,} \tag{2.11}$$

is such that

1. $M(\omega) = \sum_\ell \mu_\ell e^{i\omega\ell}$;
2. $\sum_\ell |\mu_\ell| \, |\ell|^\eta < \infty$, for some $\eta > 0$;
3. $\sup_{\omega \in \mathbb{R}} |M(\omega)| < 2^{N-1}$;

then the sequence of functions f_n defined by Eqn. (2.5) with $f_0 := \chi_{[0,1)}$[4] converges pointwise to a continuous function f_∞ whose Fourier transform is given by

$$\hat{f}_\infty = \prod_{n=1}^{\infty} m_0(\omega/2^n).$$

Regularity: Suppose $m_0(\omega)$ factors as above in Eqn. (2.11) and that item (2) holds. If, in addition,

$$\sup_{\omega \in \mathbb{R}} |M(\omega) M(\omega/2) \cdots M(\omega/2^{n-1})| = B_n,$$

then

$$\boxed{|\hat{f}_\infty| = |\textstyle\prod_{n=1}^{\infty} m_0(\omega/2^n)| \leq C(1+|\omega|)^{-N+\log B_n/(n \log 2)}.} \tag{2.12}$$

Let $B = \inf_n(B_n)$. Then

$$\boxed{f_\infty \in C^m, \text{ where } m \text{ is the largest integer strictly smaller than } N - B - 1.} \tag{2.13}$$

[4]This is not the only f_0 that works! See, for instance, [42]

Determining the $\{g_\ell\}$: In terms of the symbol $m_0(\omega)$, the requirement that $\{\phi(\cdot - \ell)\}$ forms an *orthonormal* basis of $V = \sigma[\phi]$ reads

$$\boxed{|m_0(\omega)|^2 + |m_0(\omega + \pi)|^2 = 1.} \tag{2.14}$$

Now suppose that the set of $\{g_\ell\}$ is *finite*, say $\ell = 0, 1, \ldots, L$. Then $m_0(\omega)$ is a *trigonometric polynomial* in sin and cos and, since we require some regularity, this polynomial should also satisfy Eqn. (2.11). Set $\mathcal{M}(\omega) := |m_0(\omega)|^2$ and $\mathcal{N}(\omega) := |M(\omega)|^2$. (This is also a polynomial in $\cos\omega$.)

Objective: We need to find a polynomial $\mathcal{M}(\omega)$ such that

$$\mathcal{M}(\omega) + \mathcal{M}(\omega + \pi) = 1 \tag{2.15}$$

and

$$\mathcal{M}(\omega) = \left(\cos^2 \frac{\omega}{2}\right)^N \mathcal{N}(\omega).$$

Write $\mathcal{N}(\omega)$ as a polynomial in $\sin^2 \omega/2 = (1 - \cos\omega)/2$:

$$\mathcal{N}(\omega) = P(\sin^2 \omega/2).$$

Set $x := \sin^2 \omega/2$. Then Eqn. (2.15) becomes

$$\boxed{(1 - x)^N P(x) + x^N P(1 - x) = 1.} \tag{2.16}$$

The solutions of Eqn. (2.16) determine the coefficients $\{g_\ell\}$!

By *Bézout's Theorem* there exist solutions to Eqn. (2.16).

For each $N \geq 1$, there exists an associated finite set of coefficients $\{g_\ell\}$. These define the *Daubechies' family of scaling functions* ${}_N\phi$.

Wavelets: Let $h_\ell := (-1)^\ell g_{1-\ell}$. Then the function

$$\boxed{\psi(x) := \textstyle\sum_\ell (-1)^\ell g_{1-\ell} \phi(2x - \ell)} \tag{2.17}$$

defines an orthonormal basis for $W_1 = V_1 \ominus V_0$; thus the functions $\{\psi_{k\ell}\}_{k,\ell \in \mathbb{Z}}$ form an orthonormal basis of $L^2(\mathbb{R})$.

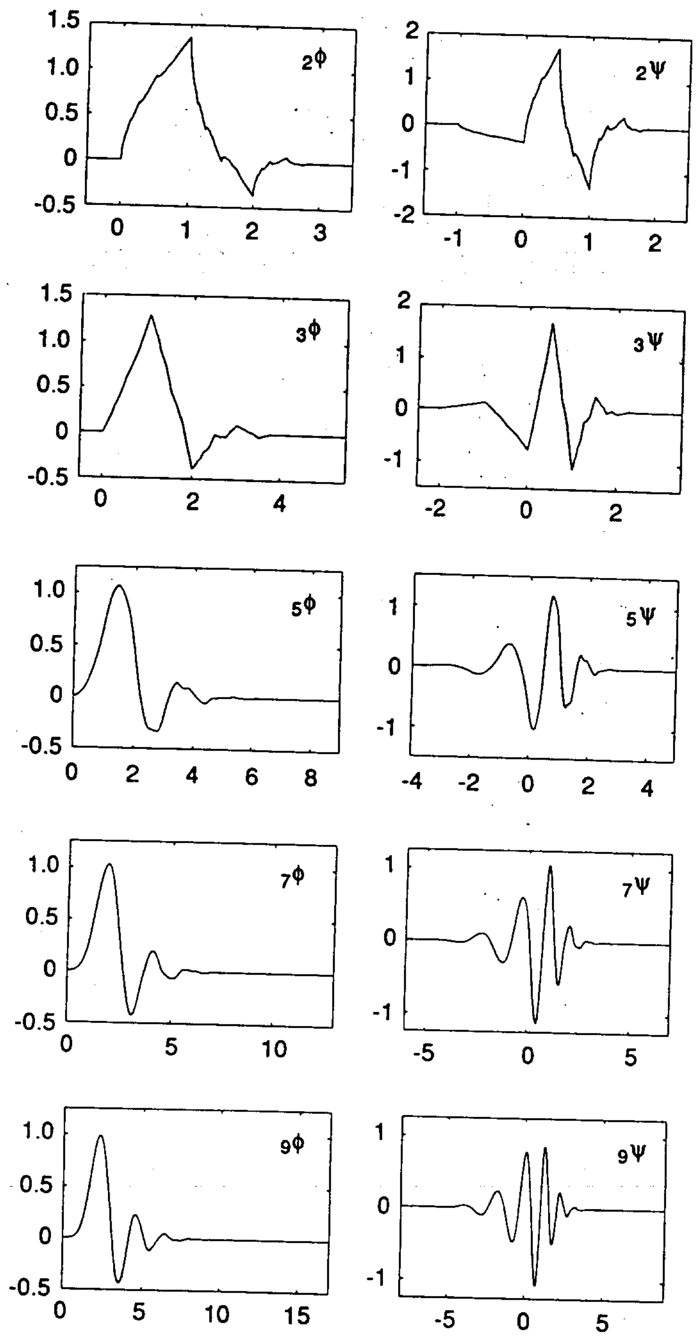


FIG. 6.3. *Plots of the scaling functions* $_N\phi$ *and wavelets* $_N\psi$ *for the compactly supported wavelets with maximum number of vanishing moments for their support width, and with the extremal phase choice, for* $N = 2, 3, 5, 7,$ *and* 9.

I. Daubechies: "Ten Lectures on Wavelets"
SIAM CBMS Vol. 61, 1992.

Properties of the Daubechies' Wavelets:

- $\phi(x) = \sum_{\ell=0}^{2N-1} g_\ell \phi(2x-\ell)$, $g_0 \neq 0 \neq g_{2N-1}$, implies that $\text{supp}\phi = [0, 2N-1]$.
- The higher the regularity the larger the support.
- If $\psi \in C^m$ and $\psi^{(m)}$ is bounded, then ψ has $m+1$ vanishing moments:

$$\boxed{\int_{\mathbf{R}} x^p \psi(x)dx = 0, \qquad p = 0, 1, \ldots, m} \tag{2.18}$$

  and ϕ reproduces polynomials of degree at most m:

$$\boxed{x^p = \sum_\ell a_\ell \phi(x-\ell), \quad \text{for some real coefficients } a_\ell.} \tag{2.19}$$

- Good localization in time and frequency domain.
- *Only* the Haar scaling function and wavelet are symmetric.
- (Unmodified) Daubechies scaling functions and wavelets show boundary effects when restricted to subsets of the real line, e.g. finite or semi-infinite intervals.
- Daubechies scaling functions and wavelets are not interpolatory. [5]
- The Daubechies wavelet ${}_2\psi$ is an *affine fractal function* [6] of the type considered later in these lectures.
- The Haar scaling function and associated wavelet are included in the Daubechies family if $N = 1$.
- There are so-called *generalized functions* or *distributions* which satisfy two-scale dilation equations but are *not* included in the Daubechies family. One such example is the *Dirac δ "function"*. It satisfies the dilation equation

$$\boxed{\delta(x) = 2\delta(2x).}$$

[5]Given a finite set of interpolation points $\{(x_j, y_j) \mid j = 0, 1, \ldots, J\}$, there exists a finite set of constants $\{c(k\ell)\}$ such that $y_j = \sum_\ell c(k\ell)\phi(2^k x_j - \ell)$, for all $j = 0, 1, \ldots, J$.

[6]An affine fractal function is a function whose graph is made up of a finite number of affine images of itself. The graph of such a function has, in general, non-integral dimension.

Cardinal Spline Wavelets (cf. [7])

The *mth order fundamental cardinal spline* L_m has the interpolation property with respect to the data $\{(i,\delta_i)\}_{i\in\mathbb{Z}}$, i.e.,

$$\boxed{L_m(i) = \delta_i := \begin{cases} 1 & \text{if } i = 0, \\ 0 & \text{otherwise} \end{cases}} \tag{2.20}$$

It is known that L_m can be written in the form

$$\boxed{L_m(x) = \textstyle\sum_{\ell\in\mathbb{Z}} c_{m,\ell} N_m\left(x + \frac{m}{2} - \ell\right),} \tag{2.21}$$

for some bi-infinite sequence of real coefficients $\{c_{m,\ell}\}$. (This bi-infinite sequence is determined by solving $L_m(i) = \delta_i$.) Note that L_m is *not* compactly supported!

Theorem 2.1 (Cardinal Spline Wavelets) *Let $m \in \mathbb{N}$ be fixed and let $W^m_{k+1} := V^m_{k+1} \ominus V^m_k$ (cf. Eqn (1.4)). Define*

$$\boxed{\psi^m(x) := L^{(m)}_{2m}(2x-1), \qquad m \in \mathbb{N}.} \tag{2.22}$$

Then the wavelet spaces $\{W^m_k\}_{k\in\mathbb{Z}}$ are generated by the $\{\psi^m_{k\ell}\}$:

$$\boxed{W^m_k = \sigma[\psi^m(2^k\,\cdot)].} \tag{2.23}$$

Note that since L_{2m} is not compactly supported, the wavelets ψ^m are not orthogonal to their translates.

Theorem 2.2 (Compactly Supported Spline Wavelets) *Let $m \in \mathbb{N}$ be fixed. The functions*

$$\boxed{\psi^m(x) := \textstyle\sum_{\ell=0}^{3m-2}\left[\frac{(-1)^n}{2^{m-1}}\sum_{n=0}^{\ell}\binom{m}{n} N_{2m}(\ell+1-n)\right] N_{2m}(2x-\ell),} \tag{2.24}$$

form a Riesz basis of W^m_0. Moreover, $\operatorname{supp}\psi^m = [0, 2m-1]$.

3 Brief Introduction to Fractal Functions

Let $\{(x_j := j/2, y_j) \mid j = 0,1,2\}$ be a given set of interpolation points. Let f_1 be the unique linear function satisfying

$$f_1(j/2) = y_j \qquad j = 0,1,2.$$

Denote by S_k^1 the vector space of all linear splines with *knots* at $\{j/2^k \mid j = 0,1,\ldots,2^k\}$.

Note that $S_k^1 \subseteq S_{k+1}^1$, $k \in \mathbb{N}$.

Define an operator $T : S_1^1 \to S_2^1$ by

$$\boxed{(Tf_1)(x) = \begin{cases} \lambda_0(2x) + s_0 f_1(2x), & 0 \le x \le 1/2 \\ \lambda_1(2x-1) + s_1 f_1(2x-1), & 1/2 < x \le 1 \end{cases},} \tag{3.1}$$

where λ_ℓ, $\ell = 0,1$, is the unique affine function such that

$$(Tf_1)(0) = f_1(0), \quad (Tf_1)(1) = f_1(1), \quad (Tf_1)(1/2-) = (Tf_1)(1/2+) = f_1(1/2).$$

(Join-up Conditions)

The s_ℓ, $\ell = 0,1$, are *free parameters.* Note that Tf_1 is a linear spline with knots $\{0, 1/4, 1/2, 3/4, 1\}$.

The iterates of B applied to f_1 generates linear splines with increasing knot sets:

$$f_{k+1} := T^k f_1 = T(T^k f_1) \in S_{k+1}^1, \qquad k \in \mathbb{N}. \tag{3.2}$$

Exercise : *Calculate f_2 explicitly for $y_0 = 0$, $y_1 = 1$, and $y_2 = 0.5$!*

Convergence of f_k as $k \to \infty$: If $\max\{|s_0, s_1\} < 1$ then f_k converges to a *continuous* function f as $k \to \infty$: Let g and g be two linear splines in S_k^1.

$$\begin{aligned} |(Tg)(x) - (Th)(x)| &\le \max\{s_0, s_1\}|f(2x-\ell) - g(2x-\ell)| \\ &\le \max\{s_0, s_1\}\sup\{|f(x) - g(x)| \mid 0 \le x \le 1\}. \end{aligned}$$

Setting $\|f\|_\infty := \sup\{|f(x)| \mid 0 \le x \le 1\}$, one has

$$\|Tg - Th\|_\infty \le \max\{s_0, s_1\}\|f - g\|_\infty. \tag{3.3}$$

Thus, if $\max\{s_0, s_1\} < 1$ the operator T is a *contraction* on the linear space of continuous functions defined on $[0,1]$. By the Banach Fixed Point Theorem,[7] T has a unique fixed point f. Moreover, f is the limit (in the $\|\cdot\|_\infty$-norm) of the sequence f_k as $k \to \infty$.

Definition 3.1 (Affine Fractal (Interpolation) Function) *The unique fixed point of the operator T defined in Eqn. (3.1) is called an affine fractal (interpolation) function. (cf. [3, 30])*

Self-referential structure of a fractal function: Let f be the affine fractal function generated by T.

$$\boxed{Tf = f \iff f(x) = \Lambda(x) + \textstyle\sum_{\ell=0}^{1} s_\ell f(2x - \ell).} \tag{3.4}$$

Here

$$\Lambda(x) := \begin{cases} \lambda_0(2x), & 0 \le x \le 1/2 \\ \lambda_1(2x-1), & 1/2 < x \le 1. \\ 0, & \text{otherwise} \end{cases}$$

and f was set to be identically zero outside $[0,1]$.

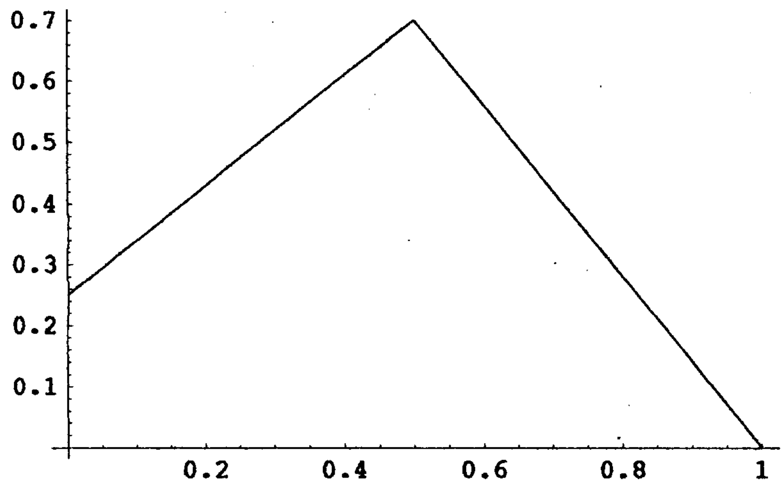
Eqn. (3.4) expresses the fact that the graph G_f of f is made up of two affine images of itself, each of which is made up of two affine images of itself, each of which is ... ad infinitum!

Exercise : *Show that $G_f = w_0(G_f) \cup w_1(G_f)$, where $w_\ell(x,y) := ((x - \ell)/2, \lambda_\ell(2x - \ell) + s_\ell y)$, $\ell = 0, 1$!*

Note that $f(j/2) = y_j$, $j = 0,1,2$, and, moreover, $f(j/2^k) = f_k(j/2^k)$, $j = 0, 1, \ldots, 2^k$, $k \in \mathbb{N}$. In other words, each space S_k^1 is properly contained in the space $\mathcal{F}^1$ of all affine fractal functions on $[0,1]$: Formally

$$\boxed{S_k^1 \stackrel{k\to\infty}{\longrightarrow} \mathcal{F}^1}$$

[7]Banach Fixed Point Theorem/Contraction Mapping Theorem: Let $(X, \|\cdot\|)$ be a complete normed linear space and let T be a contraction on X, i.e., $\|Tx - Ty\| \le C\|x - y\|$, $0 < C < 1$, then T has a unique fixed point x^* in X. Furthermore, if x_1 is *any* point in X, the sequence $\{T^k x_1\}$ converges to x^*.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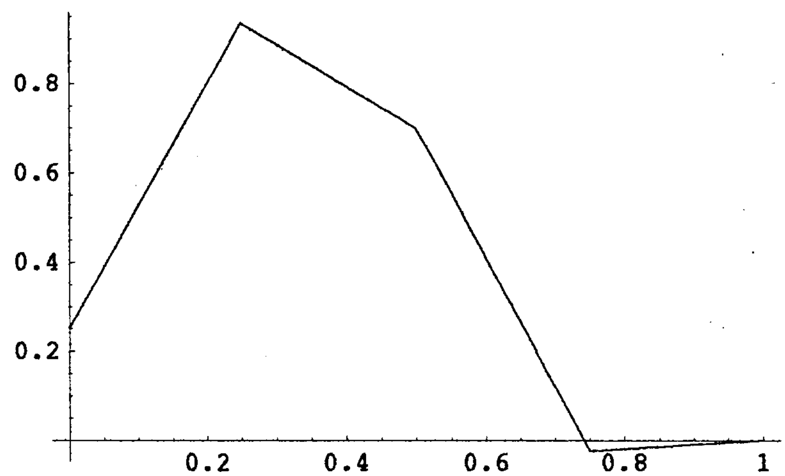
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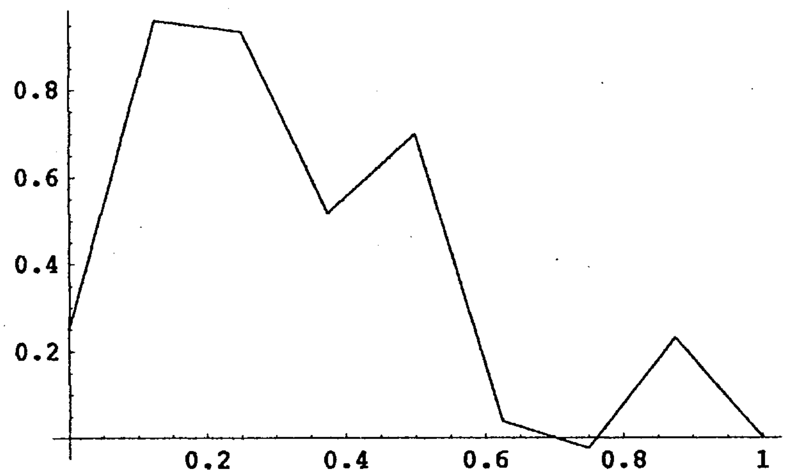
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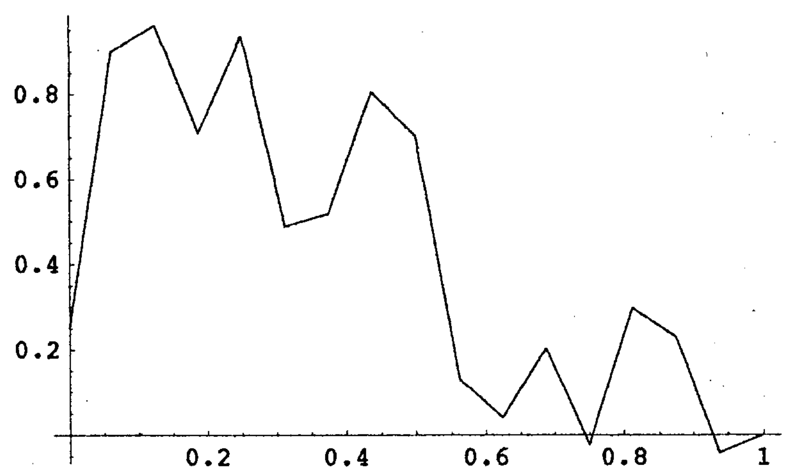
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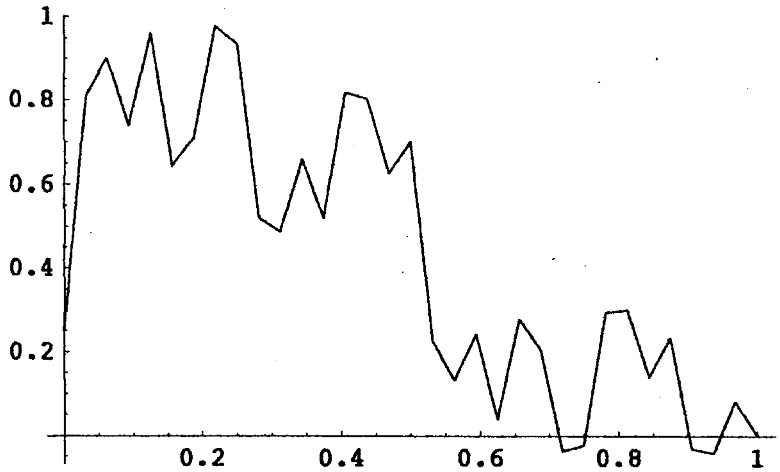
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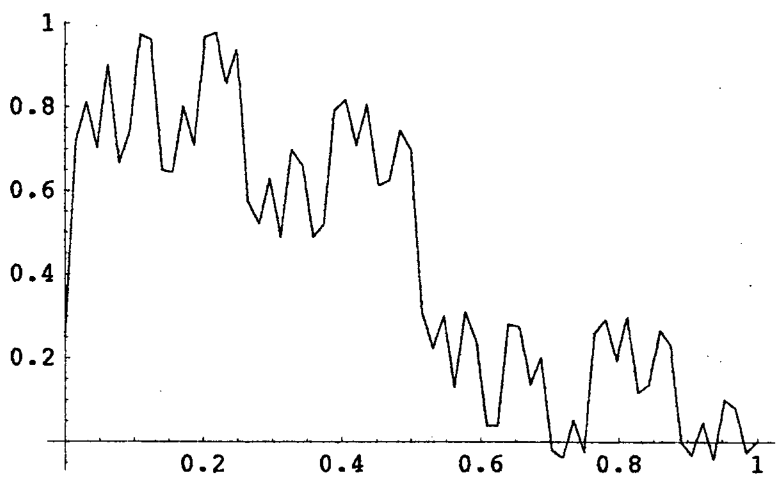
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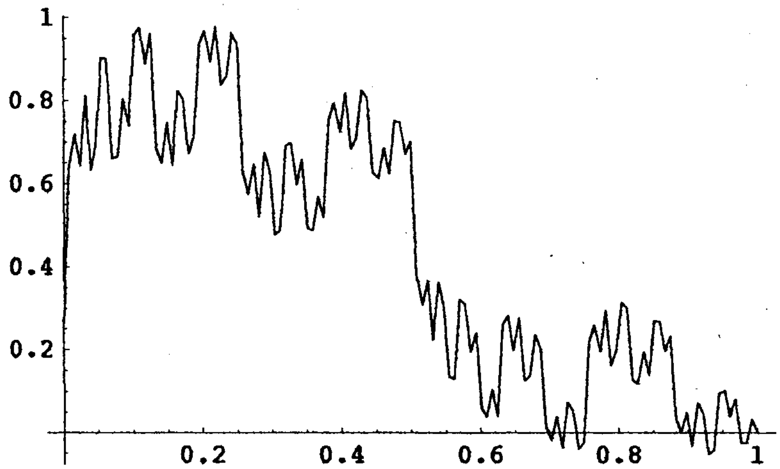
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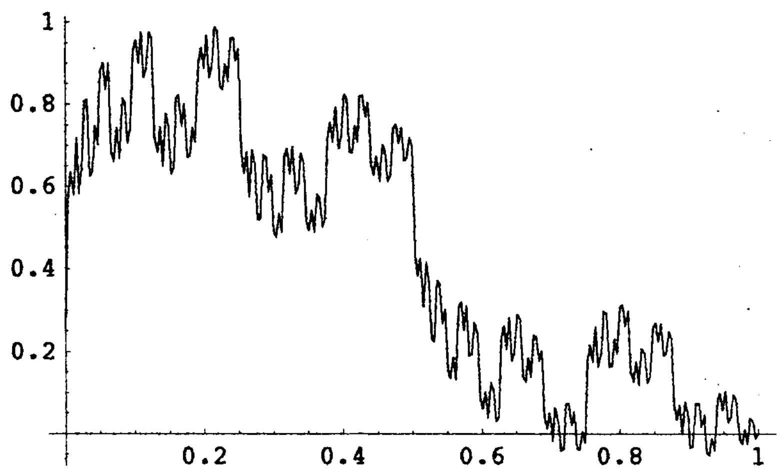
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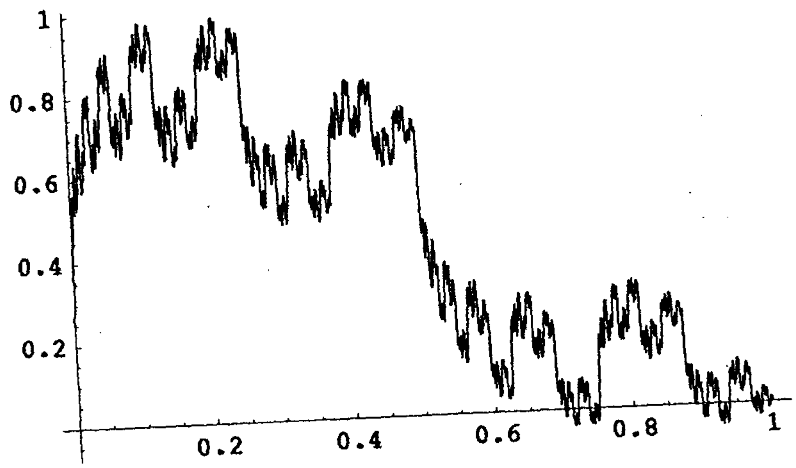
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Piecewise fractal functions

Observations:

- The interpolation values $\boldsymbol{y} := \{y_j \mid j = 0,1,2\}$, uniquely determine the affine functions $\boldsymbol{\lambda} := \{\lambda_\ell \mid \ell = 0,1\}$, which uniquely determine the affine fractal function $f = f_{(\boldsymbol{y};\boldsymbol{s})}$ for a given set of $\boldsymbol{s} := \{s_\ell \mid \ell = 0,1\}$:

$$\boxed{\boldsymbol{y} \mapsto \boldsymbol{\lambda} \mapsto f_{(\boldsymbol{y};\boldsymbol{s})}}$$

  Thus, the mapping $\boldsymbol{y} \mapsto f_{(\boldsymbol{y};\boldsymbol{s})}$ is a linear isomorphism.

- Eqn. (3.4) is an *inhomogeneous two-scale dilation equation.*

- The affine fractal function $f(\cdot/2)$ restricted to the interval $[0,1/2)$, respectively, $[1/2,1]$ is an affine fractal function on its own right generated by affine mappings $\lambda_{00} = \lambda_0$, $\lambda_{01} = \lambda_0(\cdot - 1) + s_1\lambda_1 - s_0\lambda_0$ on $[0,1/2)$, respectively by $\lambda_{10} = \lambda_1 + s_0\lambda_0 - s_1\lambda_1$ and $\lambda_{11} = \lambda_1(\cdot - 1)$ on $[1/2,1]$.

As $f = f_{(\boldsymbol{y};\boldsymbol{s})}$ is uniquely determined by $\boldsymbol{y}$, there exists a *canonical basis* for $\mathcal{F}^1$: Fix a set of parameters $\{s_\ell \mid \ell = 0,1\}$ and let $\boldsymbol{y}_0 := \{1,0,0\}$, $\boldsymbol{y}_1 := \{0,1,0\}$, and $\boldsymbol{y}_2 := \{0,0,1\}$. Let ϵ_j be the unique affine fractal function generated by the interpolation values $\boldsymbol{y}_j$, $j = 0,1,2$. If $\boldsymbol{y} = \{y_j \mid j = 0,1,2\}$ is any set of interpolation values and $f = f_{\boldsymbol{y}}$ the uniquely determined fractal function, then

$$\boxed{f_{\boldsymbol{y}} = \textstyle\sum_{j=0}^{2} y_j\,\epsilon_j.} \tag{3.5}$$

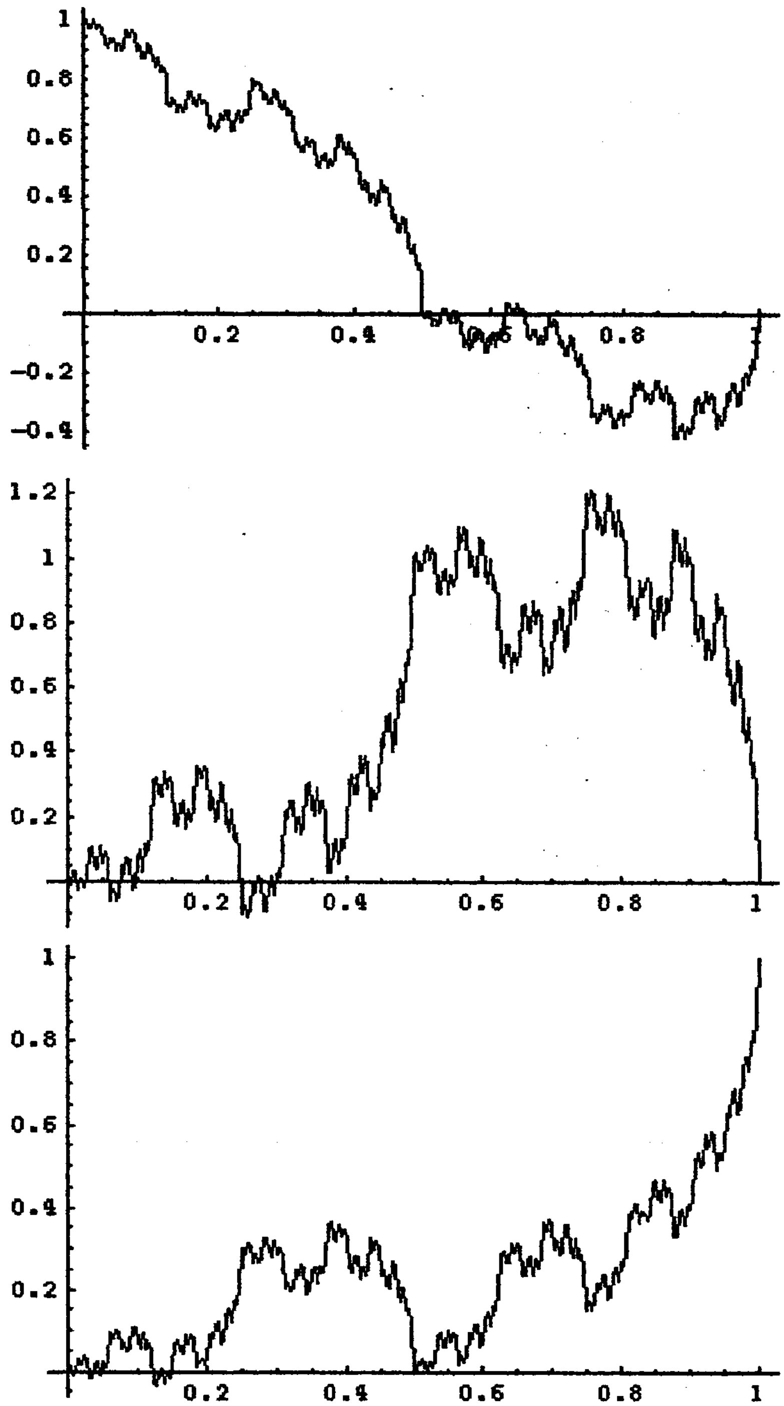

Figure 2: The canonical basis ϵ_j.

Definition 3.2 (Piecewise Fractal Function) *On each interval of the form $[\ell, \ell+1)$ construct a fractal function f_ℓ, $\ell \in \mathbb{Z}$. Then the function*

$$\boxed{f = \sum_{\ell=-\infty}^{+\infty} f_\ell \chi_{[\ell,\ell+1)}} \tag{3.6}$$

is called a piecewise fractal function.

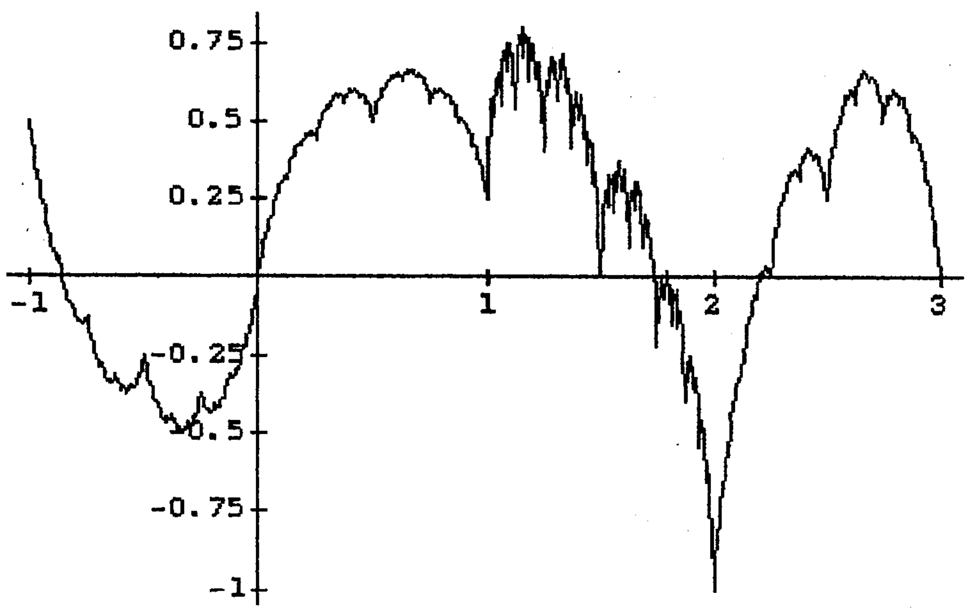

Figure 3: A piecewise fractal function

Finitely generated shift-invariant and refinable spaces may be constructed using these piecewise fractal functions: For instance, is shown shortly that the space

$$\boxed{V_0 := \{g : \mathbb{R} \to \mathbb{R} \mid g|_{[\ell,\ell+1)} = f_\ell, \ell \in \mathbb{Z}\} \cap L^2(\mathbb{R})} \tag{3.7}$$

is spanned by two *orthogonal* generators. Moreover, the refinable spaces $\{V_k\}_{k \in \mathbb{Z}}$ form an *orthogonal MRA of $L^2(\mathbb{R})$*.

4 Construction of the DGHM Multiwavelet

(Presentation given here is due to Donovan, Geronimo, and Hardin)

Recall Eqn. (1.10). Start again with the hat function $h(x) = (1 - |x|)^+$. Introduce a new and yet unknown continuous function w with supp$w = [0, 1]$, and define $V_0 := \sigma[h, w]$.

Idea: Find a function u such that

- u is supported on $[-1, 1]$;
- u is a linear combination of h, w, and $w(\cdot + 1)$;
- u is orthogonal to its translates $u(\cdot \pm 1)$ and to w;
- $V_0 = \sigma[u, w]$.

Define

$$\begin{aligned} u(x) &:= (I - P_{\sigma[w]})h \\ &= h - \frac{\langle h, w\rangle}{\langle w, w\rangle} w - \frac{\langle h, w(\cdot + 1)\rangle}{\langle w, w\rangle} w(\cdot + 1). \end{aligned}$$

Note that w is already orthogonal to its integer shifts. Need

$$\langle u, u \cdot -1\rangle = 0.$$

This is equivalent to

$$\boxed{\langle h, h(\cdot - 1)\rangle = \tfrac{\langle u,w\rangle \langle u(\cdot - 1), w\rangle}{\langle w,w\rangle}.} \tag{4.1}$$

Refinability implies that $w(\cdot/2) \in V_0$; i.e., $w(\cdot/2)$ must be a linear combination of $h(\cdot - 1)$, w, and $w(\cdot - 1)$:

$$\boxed{w(x/2) = h(x-1) + s_0 w(x) + s_1 w(x-1).} \tag{4.2}$$

But Eqn. (4.2) is recognized as an inhomogenous two-scale dilation equation of the form (3.4)! Thus, if $\max\{|s_0|, |s_1|\} < 1$, the solution of Eqn. (4.2) is an affine fractal function. (Note that $\lambda_0(x) = x$ and $\lambda_1(x) = 1 - x$.)

Employing the fixed point equation (3.4), one can derive explicit formulas for the inner products of fractal functions.

Exercise : *Suppose that f and g are two affine fractal functions generated by affine map λ_ℓ, respectively, μ_ℓ, $\ell = 0, 1$. Use Eqn. (3.4) to derive an explicit formula for the inner product $\langle f, g\rangle$!*

Choosing $s_0 = s_1 =: s$ causes w to be symmetric about the line $x = 1/2$. (Verify this!). In this case, one obtains

$$\begin{aligned}
\langle w, 1\rangle &= \frac{1}{2(1-s)} \\
\langle h, w\rangle &= \langle h, w(\cdot + 1)\rangle = \frac{1}{4(1-s)} \\
\langle w, w\rangle &= \frac{2+s}{6(1-s)^2(1+s)} \\
\langle h, h(\cdot - 1)\rangle &= \frac{1}{6}.
\end{aligned}$$

The orthogonality condition (4.1) then gives

$$\boxed{s = -1/5}$$

Normalizing u and w yields:

$$\begin{aligned}
\phi^1(x) &:= \frac{w}{\sqrt{\langle w, w\rangle}} \\
\phi^2(x) &:= \frac{u}{\sqrt{\langle u, u\rangle}}.
\end{aligned}$$

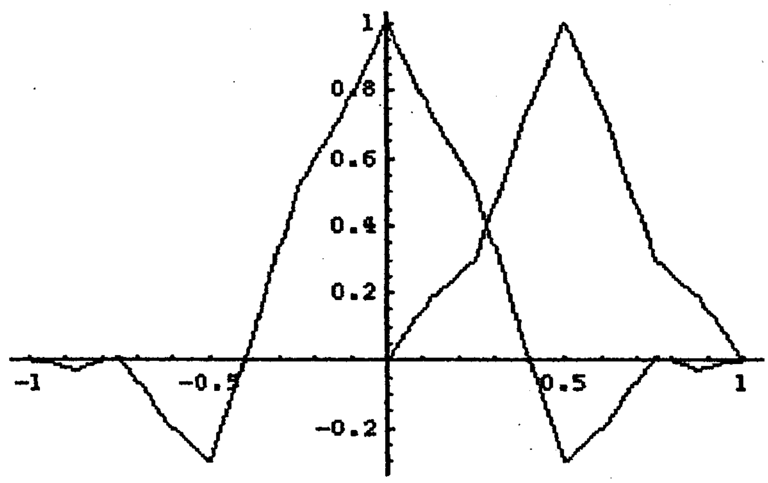

Figure 4: The orthogonal generators ϕ^1 and ϕ^2.

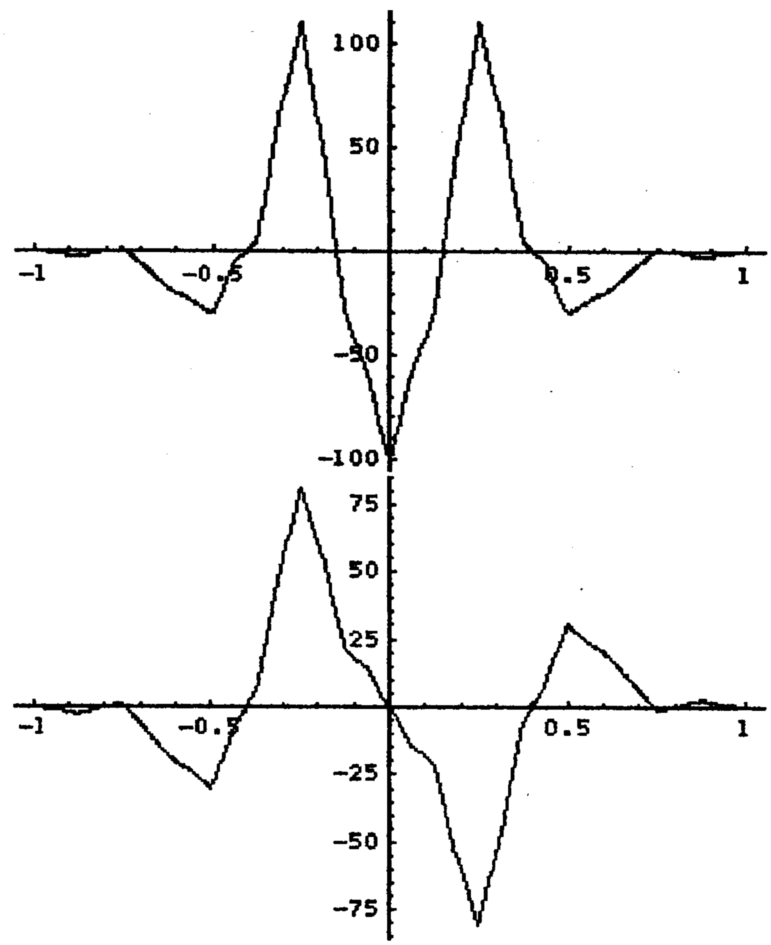

Figure 5: The orthogonal multiwavelet ψ^1 and ψ^2.

Properties of the DGHM scaling vector:

- ϕ^2 is made up of two affine fractal functions; one with support $[-1,0]$ the other with support $[0,1]$.
- Since $|s| = 1/5 < 1/2$, the orthogonal generators are elements of the function space Lip^1, [8] and thus possess a first derivative almost everywhere. This first derivative is an element of $L^2(\mathbb{R})$.
- Since the hat function is a linear combination of ϕ^1 and ϕ^2 (Show this!), the scaling vector $\boldsymbol{\phi} := (\phi^1\phi^2)^T$ has the same approximation order as the hat function:

$$x^p = \sum_\ell c_\ell^T \boldsymbol{\phi}(x-\ell), \qquad p = 0, 1.$$

- The dilates and translates $\boldsymbol{\phi}_{k\ell}$ *remain* orthogonal when restricted to compact intervals. (This is very important when (multi) wavelets are employed to solve boundary value problems.)
- $\boldsymbol{\phi}$ can handle *non-uniform geometries* such as irregular grid spacings.
- The scaling vector $\boldsymbol{\phi}$ has smaller support and higher regularity than the corresponding Daubechies scaling function with the same approximation order, namely, ${}_2\phi$.
- Symmetry/antisymmetry
- $\boldsymbol{\phi}$ is interpolatory (since ϕ^1 and ϕ^2 are fractal interpolation functions).

[8] A function f is said to belong to the function space Lip^1 if there exists a positive constant C such that

$$|f(x) - f(x')| < C\,|x - x'| \text{ for all } x, x' \in \mathbb{R}.$$

Calculation of Matrix Coefficients for the Scaling Vector:

Analytically

Matrix Refinement Equation:

$$\boxed{\begin{pmatrix} \phi^1(x) \\ \phi^2(x) \end{pmatrix} = \sqrt{2} \textstyle\sum_{\ell=-2}^{1} G_\ell \begin{pmatrix} \phi^1(2x-\ell) \\ \phi^2(2x-\ell) \end{pmatrix}.} \tag{4.3}$$

The values of $\boldsymbol{\phi} = (\phi^1\phi^2)^T$ at $x_j = j/2^2$, $j = 1, \ldots, 2^3$, are known. (Fixed Point Equation for fractal functions!) Thus, the entries in the matrices G_ℓ, $\ell = -2, \ldots, 1$, can be computed from Eqn. (4.3).

$$\boxed{\begin{array}{llll} G_{-2} = \begin{pmatrix} 3\sqrt{2}/10 & 4/5 \\ -1/20 & -3\sqrt{2}/20 \end{pmatrix}, & G_{-1} = \begin{pmatrix} 3\sqrt{2}/10 & 0 \\ 9/20 & 1/\sqrt{2} \end{pmatrix} \\ G_0 = \begin{pmatrix} 0 & 0 \\ 9/29 & -3\sqrt{2}/20 \end{pmatrix}, & G_1 = \begin{pmatrix} 0 & 0 \\ -1/20 & 0 \end{pmatrix} \end{array}}$$

Geometrically (Donovan-Geronimo-Hardin)

- $\boldsymbol{\alpha} := (abc)^T$, $\boldsymbol{\beta} := (cba)^T$, and $\boldsymbol{\gamma} := (def)^T$;
- $\langle \phi^2, \phi^2 \rangle = \|\boldsymbol{\alpha}\|^2 + (1/\sqrt{2})^2 + \|\boldsymbol{\beta}\|^2 = 1$;
- $\|\boldsymbol{\alpha}\|^2 = \|\boldsymbol{\beta}\|^2$ and $e = \phi^1(1/2)$;
- $\langle \phi^i(\cdot - \ell), \phi^j \rangle = \delta_{ij}\delta_\ell$. This is equivalent to saying that $\{\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}\}$ is an orthonormal basis of $\mathbb{R}^3$.

Calculation of the Multiwavelet

Analytically

Solve Eqns. (1)– (1.18)

$$\boxed{\begin{array}{llll} H_{-2} = \begin{pmatrix} \sqrt{3}/20 & 3\sqrt{6}/20 \\ 0 & 0 \end{pmatrix}, & H_{-1} = \begin{pmatrix} -9\sqrt{3}/20 & 1/\sqrt{6} \\ 0 & -1/\sqrt{3} \end{pmatrix} \\ H_0 = \begin{pmatrix} 3\sqrt{3}/20 & -\sqrt{6}/20 \\ 3\sqrt{6}/10 & -\sqrt{35} \end{pmatrix}, & H_1 = \begin{pmatrix} -\sqrt{3}/60 & 0 \\ -\sqrt{6}/30 & 0 \end{pmatrix} \end{array}}$$

Geometrically (Donovan-Geronimo-Hardin)

- Look for two wavelets supported on $[-1, 1]$;
- Let $\boldsymbol{\delta} := (a'b'c')^T$ and $\boldsymbol{\varepsilon} := (d'e'f')^T$;
- $\langle \psi, \phi^1 \rangle = \langle \psi, \phi^2(\cdot - 1) \rangle = 0$ implies $\boldsymbol{\varepsilon} \bullet \boldsymbol{\gamma} = \boldsymbol{\varepsilon} \bullet \boldsymbol{\alpha} = 0$.
  $\Longrightarrow$ $\boldsymbol{\varepsilon}$ multiple of $\boldsymbol{\beta}$.
  Similarly, $\boldsymbol{\delta}$ multiple of $\boldsymbol{\alpha}$.
- $\psi^1 : (\boldsymbol{\alpha})\,(h)\,(\boldsymbol{\beta})$ (symmetry);
- $\psi^2 : (-\boldsymbol{\alpha})\,(0)\,(\boldsymbol{\beta})$ (antisymmetry);
- $\langle \psi^1, \phi^2 \rangle \Longrightarrow h = -1/\sqrt{2}$;

More on Scaling Vectors and Multiwavelets

Intertwining MRAs

Let ϕ be a finite collection of compactly supported L^2-functions generating an MRA:

$$V_k := \sigma[\phi(2^k \cdot)] \qquad k \in \mathbb{Z}.$$

Theorem 4.1 (Donovan-Geronimo-Hardin) *There exists a pair of integers* (k, m) *and some* orthogonal *MRA* $\{\widetilde{V}_m\}_{m \in \mathbb{Z}}$ *such that*

$$\boxed{V_k \subset \widetilde{V}_0 \subset V_{k+m}.} \tag{4.4}$$

$\{V_k\}$ and $\{\widetilde{V}_m\}$ are called *intertwining MRAs.*

Intertwining MRAs may be used to construct an ***orthogonal*** MRA from a nonorthogonal. The basic idea is to use some of the generators from V_{k+m} to modify the generators of $\widetilde{V}_0$.

Example 4.1 *Piecewise Quadratic Orthonormal Scaling Vector (cf. [38])*

Let $h = (1 - |x|)^+$ be the hat function, let $q(x) := (4x - 4x^2)^+$, and let $V_0 := \sigma[h, q]$. It follows from approximation-theoretic considerations that the refinable space V_0 has approximation order three, i.e., every polynomial of degree at most two can be written as a linear combination of the translates of h and q. Clearly, h and q are *not* orthogonal generators.

Objective: Find an orthonormal refinable subspace $\widetilde{V}_0$ of V_0 such that

$$V_0 \supset \widetilde{V}_0 \supset V_1 \supset \widetilde{V}_1.$$

For this purpose, let

$$\begin{aligned} h_1(x) &:= (I - P_{\sigma[q]})h(x) \\ &= h(x) - \frac{\langle q, h\rangle}{\langle q, q\rangle} q(x) - \frac{\langle q(\cdot+1), h\rangle}{\langle q, q\rangle} q(x+1). \end{aligned}$$

Note: $q \perp h_1$.

However, $\langle h_1, h_1(\cdot - 1)\rangle \neq 0$.

Now choose a function h_2 with support $[0,1]$ from V_1. Note that functions from V_1 restricted to $[0,1]$ form a three dimensional space $\sigma[h(2\cdot -1), q(2\cdot), q(2\cdot -1)]$.

$$h_3(x) := (I - P_{\sigma[h_2]})h_1(x).$$

Need:

- $\langle h_2, q\rangle = 0 \Longrightarrow h_2 \in q^\perp \cap V_1$.
- $\langle h_3, h_3(\cdot - 1)\rangle = 0$.

The first requirement is easily satisfied: As $q^\perp \cap V_1$ is two dimensional let

$$\begin{aligned} p_1(x) &:= q(2x) - q(2x-1), \quad \text{antisymmetric about } x = 1/2! \\ p_2(x) &:= q(2x) + q(2x-1) - \frac{28}{25}h(2x-1). \end{aligned}$$

Thus,

$$h_2(x) = s_1 p_1(x) + s_2 p_2(x).$$

Now

$$\langle h_3, h_3(\cdot - 1)\rangle = \langle h_1, h_1(\cdot - 1)\rangle - \frac{\langle h_1, h_2\rangle\langle h_1(\cdot - 1), h_2\rangle}{\langle h_2, h_2\rangle}. \quad \text{(Show this!)}. \tag{4.5}$$

Thus,

$$\boxed{\langle h_1, h_1(\cdot - 1)\rangle = \frac{\langle h_1,h_2\rangle\langle h_1(\cdot -1),h_2\rangle}{\langle h_2,h_2\rangle}} \tag{4.6}$$

The unknowns s_1 and s_2 are now determined by Eqn. (4.5):

$$\boxed{125 s_1^2 - 256 s_2^2 = 0} \tag{4.7}$$

Normalization yields:

$$\boxed{\begin{aligned} \phi^1(x) &:= \frac{q(x)}{\|q\|} \\ \phi^2(x) &:= \frac{h_2(x)}{\|h_2\|} \\ \phi^3(x) &:= \frac{h_3(x)}{\|h_3\|} \end{aligned}}$$

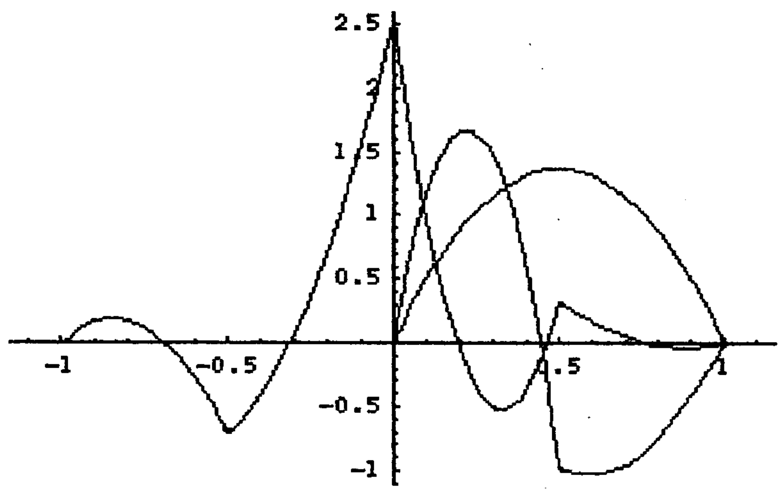

Figure 6: The orthonormal generators ϕ^1, ϕ^2, and ϕ^3.

Support Properties of Scaling Vectors

Recall that if ϕ is a scaling function whose refinement equation is

$$\phi(x) = \sum_{\ell=0}^{L} g_\ell \phi(2x - \ell), \quad g_0 g_L \neq 0,$$

then $\text{supp}\phi = [0, L]$.

The situation is not as simple for scaling vectors.

Theorem 4.2 *Suppose the scaling vector* $\boldsymbol{\phi}$ *satisfies*

$$\boldsymbol{\phi}(x) = \sum_{\ell=0}^{L} G_\ell \boldsymbol{\phi}(2x - \ell)$$

with $G_0, G_L \neq O \in \mathbb{R}^{r \times r}$. *If*

1. *If* G_0 *is nilpotent* [9] *then* $\text{supp}\boldsymbol{\phi} \subseteq [1/(2^r - 1), L]$*;*
2. *If* G_L *is nilpotent then* $\text{supp}\boldsymbol{\phi} \subseteq [0, L - 1/(2^r - 1)]$*;*
3. *If neither* G_0 *nor* G_L *is nilpotent then* $\text{supp}\boldsymbol{\phi} \doteq [0, L]$.

Remark: Tighter bounds can be obtained by considering the individual entries in the matrices G_ℓ. (cf. [39]).

[9] A matrix M is called nilpotent if some positive integer power of M is the zero matrix: there exists an $n \in \mathbb{N}$ such that $M^n = O$.

5 Applications

Decomposition and Reconstruction Algorithm

Since $V_{k+1} = V_k \oplus W_{k+1}$, every $f_{k+1} \in V_{k+1}$ can be *decomposed* into an "averaged" or "blurred" component $f_k \in V_k$ and a "difference" or "fine-structure" component $g_{k+1} \in W_{k+1}$:

$$f_{k+1} = f_k + g_{k+1}.$$

This decomposition can be continued until f_{k+1} is decomposed into a coarsest component f_0 and k difference components g_m, $m = 1, \ldots, k+1$:

$$\boxed{f_{k+1} = f_0 + g_1 + \ldots + g_k + g_{k+1}.} \tag{5.1}$$

This *decomposition algorithm* can be reversed to give a *reconstruction algorithm*: Given the coarse components together with the fine structure components one reconstructs any $f_k \in V_k$ via reversal of Eqn. (5.1).

Both algorithms are usually applied to the expansion coefficients (in terms of the underlying basis) of f and g and they involve the matrices G_ℓ and H_ℓ.

More precisely, the decomposition algorithm gives

$$\boxed{\begin{aligned} V_k \ni f_k &= \textstyle\sum_\ell \boldsymbol{\alpha}_{k\ell}^T \boldsymbol{\phi}(2^k \cdot - \ell) \\ &= \textstyle\sum_\ell \boldsymbol{\alpha}_{k-1,\ell}^T \boldsymbol{\phi}(2^{k-1} \cdot - \ell) + \sum_\ell \boldsymbol{\beta}_{k-1,\ell}^T \boldsymbol{\psi}(2^{k-1} \cdot - \ell), \end{aligned}}$$

where the *vector coefficients* $\boldsymbol{\alpha}_{k\ell}$, $\boldsymbol{\alpha}_{k-1,\ell}$, and $\boldsymbol{\beta}_{k-1,\ell}$ are related via

$$\boxed{\boldsymbol{\alpha}_{k-1,\ell} = \textstyle\sum_{\ell'} C_{\ell'-2\ell}\, \boldsymbol{\alpha}_{k\ell} \quad \text{and} \quad \boldsymbol{\beta}_{k-1,\ell} = \sum_{\ell'} D_{\ell'-2\ell}\, \boldsymbol{\alpha}_{k\ell}.} \tag{5.2}$$

This last equation defines an operator $\mathcal{D}_k : \ell^2(\mathbb{R}^r) :\to \ell^2(\mathbb{R}^r) \times \ell^2(\mathbb{R}^r)$ via

$$\boxed{\mathcal{D}_k(\boldsymbol{\alpha}_{k\ell}) = (\boldsymbol{\alpha}_{k-1,\ell}, \boldsymbol{\beta}_{k-1,\ell}), \qquad k \in \mathbb{Z},} \tag{5.3}$$

where the right-hand side is given by Eqn. (5.2).

Note that $\boldsymbol{\alpha}_{k-1}$ and $\boldsymbol{\beta}_{k-1}$ are *sampled* only at the *even* integers (down-sampling by 2: $\downarrow 2$).

$$\boxed{\begin{tikzcd}
\boldsymbol{\alpha}_0 \arrow[r] \arrow[dr] & \boldsymbol{\alpha}_1 \arrow[r] \arrow[dr] & \cdots \arrow[r] \arrow[dr] & \boldsymbol{\alpha}_{N-1} \arrow[r] \arrow[dr] & \boldsymbol{\alpha}_N \\
& \boldsymbol{\beta}_1 & \boldsymbol{\beta}_2 \cdots & \boldsymbol{\beta}_{N-1} & \boldsymbol{\beta}_N
\end{tikzcd}} \tag{5.6}$$

Figure 7: The decomposition algorithm

$$\boxed{\begin{tikzcd}
\boldsymbol{\alpha}_0 \arrow[r] & \boldsymbol{\alpha}_1 \arrow[r] & \cdots \arrow[r] & \boldsymbol{\alpha}_{N-1} \arrow[r] & \boldsymbol{\alpha}_N \\
\boldsymbol{\beta}_1 \arrow[ur] & \boldsymbol{\beta}_2 \cdots \arrow[ur] & \boldsymbol{\beta}_{N-1} \arrow[ur] & \boldsymbol{\beta}_N \arrow[ur] &
\end{tikzcd}} \tag{5.7}$$

Figure 8: The reconstruction algorithm

The reconstruction algorithm is obtained as follows: If $V_{k-1} \ni f_{k-1} = \sum_\ell \boldsymbol{\alpha}^T_{k-1,\ell}\phi(2^{k-1}\cdot - \ell)$ and $W_k \ni g_k = \sum_\ell \boldsymbol{\beta}^T_{k-1,\ell}\psi(2^{k-1}\cdot - \ell)$, then

$$\boxed{\boldsymbol{\alpha}_{k\ell} = \textstyle\sum_{\ell'} C_{\ell'-2\ell}\,\boldsymbol{\alpha}_{k-1,\ell} + D_{\ell'-2\ell}\,\boldsymbol{\beta}_{k-1,\ell}.} \tag{5.4}$$

Again, this last equation implies the existence of an operator $\mathcal{R}_{k-1} : \ell^2(\mathbb{R}^r) \times \ell^2(\mathbb{R}^r) \to \ell^2(\mathbb{R}^r)$ given by

$$\boxed{\mathcal{R}_{k-1}(\boldsymbol{\alpha}_{k-1,\ell}, \boldsymbol{\beta}_{k-1,\ell}) = \boldsymbol{\alpha}_{k\ell}, \qquad k \in \mathbb{Z},} \tag{5.5}$$

with the right-hand side given by Eqn. (5.4).

Note that only the *even* indices are used to obtain $\boldsymbol{\alpha}_k$. Zeros are used for the odd indices (interlacing of zeros or upsampling by 2: $\uparrow 2$).

Compression

The finiteness of the decomposition and reconstruction algorithm suggests the following compression schemes, also called *quantization.*

Suppose $\boldsymbol{\alpha}_N \in V_N$ has been decomposed into

$$\boldsymbol{\alpha}_N \rightarrow (\boldsymbol{\alpha}_0 \, \boldsymbol{\beta}_1 \cdots \boldsymbol{\beta}_N)$$

Choose a threshold $\tau_n > 0$, for each level $n = 1, \ldots, N$.

$$\boxed{\begin{array}{lll} \text{FOR } (n = 1 \text{ to } n = N) & \text{DO} & \\ & \text{IF} & \|\boldsymbol{\beta}_n\| < \tau_n \\ & & \text{set } \|\boldsymbol{\beta}_n\| = 0 \\ & \text{ELSE} & \text{retain} \end{array}} \tag{5.8}$$

This creates a new sequence $(\tilde{\boldsymbol{\beta}}_N \, \tilde{\boldsymbol{\beta}}_{N-1} \cdots \tilde{\boldsymbol{\beta}}_1)$.

Reconstruction:

$$\boxed{\begin{array}{ccccccccc} \boldsymbol{\alpha}_0 & \longrightarrow & \tilde{\boldsymbol{\alpha}}_1 & \longrightarrow & \cdots & \longrightarrow & \tilde{\boldsymbol{\alpha}}_{N-1} & \longrightarrow & \tilde{\boldsymbol{\alpha}}_N \\ & \nearrow & & \nearrow & & \nearrow & & \nearrow & \\ \tilde{\boldsymbol{\beta}}_1 & & \tilde{\boldsymbol{\beta}}_2 & \cdots & \tilde{\boldsymbol{\beta}}_{N-1} & & \tilde{\boldsymbol{\beta}}_N & & \end{array}} \tag{5.9}$$

Compression: $\|\boldsymbol{\alpha}_N - \tilde{\boldsymbol{\alpha}}_N\|_2 < \tau$

Compression ratio:

$$\frac{\#\boldsymbol{\alpha}}{\#\tilde{\boldsymbol{\alpha}}}$$

Signal Processing

$$\begin{array}{ccccc} & & \boxed{\downarrow 2} \longrightarrow \cdots \longrightarrow \boxed{\uparrow 2} & & \\ & \overset{G}{\nearrow} & & \overset{G^T}{\searrow} & \\ \boldsymbol{\alpha}_k \longrightarrow & & & & \longrightarrow \boldsymbol{\alpha}_k \\ & \overset{H}{\searrow} & & \overset{H^T}{\nearrow} & \\ & & \boxed{\downarrow 2} \longrightarrow \cdots \longrightarrow \boxed{\uparrow 2} & & \end{array} \tag{5.10}$$

G: low-pass filter $\qquad$ H: high-pass filter

Signal: discretely sampled function $\{f(n) \mid n \in \mathbb{Z}\}$. Usually $f \in L^2(\mathbb{R})$ (finite energy); thus, $\{f(n)\} \in \ell^2(\mathbb{R})$.

Polyphase form: $\boldsymbol{f} \in \ell^2(\mathbb{R})^r$, $r \geq 1$.

$$\boldsymbol{f}(n) := \begin{pmatrix} f(rn) \\ f(rn+1) \\ f(rn+2) \\ \vdots \\ f(rn+r-1) \end{pmatrix} \in \mathbb{R}^r. \tag{5.11}$$

Associate with $\boldsymbol{f}$ a sequence $\boldsymbol{y}$ via some linear, continuous, invertible, and time-invariant operator $\Theta : \ell^2(\mathbb{R})^r \to \ell^2(\mathbb{R})^r$:

$$\boldsymbol{y} := \Theta \boldsymbol{f}. \tag{5.12}$$

It is well-known that an operator such as Θ is a *convolution operator* (cf. [21]). I.e., there exists a bi-infinite matrix $q = (q(n))$, $q(n)$ an $r \times r$ matrix, such that

$$\boxed{\boldsymbol{y} = \Theta \boldsymbol{f} = q * \boldsymbol{f}}. \tag{5.13}$$

q or its *z-transform* $Q(z) := \sum_n q(n) z^n$ is called a *prefilter* for Θ.

If $r = 1$, the identity is commonly used for Θ.

Conditions on Prefilters:

- Orthogonality: q preserves the energy of the signal.
- Preserving approximation order: if $\boldsymbol{f}$ is a polynomial signal then $q * \boldsymbol{f}$ are also samples of a polynomial (of the same degree).

Without prefiltering, constant signals may become non-constant.

Compression schemes applied to signals

- Good reconstruction;
- Denoising (uses statistical methods). Cf. [19, 20];

Data and Image Compression

Image: Sequence of gray-scales ranging from 0 (black) to 255 (white).

$M \times N$ grayscale image $\longmapsto$ sequence of length MN (unfolding of columns/rows).

Data/Image compression: Represent the same image by a sequence $\boldsymbol{y}$ of length (considerably) less than MN.

Lossy compression: information/data is discarded, cannot be recovered.

Apply the compression scheme (5.8) to "Lena" using the DGHM multiwavelet and compare to the Daubechies $D4 = {}_2\phi$ and JPEG.

Pictures and information taken from [38].

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Lena	512x512
CompRatio	17:1
PSNR	34.98
MWAV	DGHM

Lena	512x512
CompRatio	17:1
PSNR	34.45
WAV	JPEG

Figure 16: Orthogonal Prefiltered DGHM vs JPEG: Compression Ratio 17:1

Lena	512x512
CompRatio	33:1
PSNR	31.83
MWAV	DGHM

Lena	512x512
CompRatio	33:1
PSNR	30.5226
WAV	JPEG

Figure 17: Orthogonal Prefiltered DGHM vs JPEG: Compression Ratio 33:1

Lena	512x512
CompRatio	39:1
PSNR	30.53
MWAV	DGHM

Lena	512x512
CompRatio	39:1
PSNR	28.97
WAV	JPEG

Figure 18: Orthogonal Prefiltered DGHM vs JPEG: Compression Ratio 39:1

Lena	512x512
CompRatio	60:1
PSNR	29.29
MWAV	DGHM

Lena	512x512
CompRatio	60:1
PSNR	24.4333
WAV	JPEG

Figure 19: Orthogonal Prefiltered DGHM vs JPEG: Compression Ratio 60:1

Differential Equations

$$-u" + u = f, \qquad u(0) = u(1) = 0. \tag{5.14}$$

Classical Solution: $u \in C^2$ and $f \in C$.

Requirements are too strong for some realistic problems such as shock waves, turbulence, etc.

Weak Solution: Let $v \in C^1$.

$$-\int_0^1 u" v dx + \int_0^1 uv dx = \int_0^1 fv dx$$

Integration by parts together with boundary conditions yields:

$$\boxed{\int_0^1 u' v' dx + \int_0^1 uv dx = \int_0^1 fv dx,} \tag{5.15}$$

or, equivalently,

$$\boxed{\langle u', v' \rangle + \langle, u \rangle v = \langle f, v \rangle.} \tag{5.16}$$

Requirements now are: $\boxed{u \in H_0^1[0,1] \text{ and } f \in H^{-1}}$.

$$\boxed{H_0^1[0,1] = \{f \mid f' \in L^2(\mathbb{R}); f(0) = f(1) = 0\}.}$$

$$\boxed{H^{-1} = \{\text{all linear functionals } \varphi : H_0^1 \to \mathbb{R}\}.}$$

Galerkin Method: Approximate weak infinite dimensional solution space by finite dimensional approximation space V_k such that $V_k \to H_0^1$ as $k \to \infty$.

Exercise: *Suppose $\{e_i\}_{1 \leq i \leq N}$ is a basis for some finite dimensional approximation space V_k. Project u onto V_k:*

$$u_k(x) = \sum_{i=1}^{N} c_i e_i(x), \quad c_i \in \mathbb{R}.$$

Show that taking as v in Eqn. (5.15) all the e_j, $j = 1, \ldots, N$, yields an algebraic *linear system for the unknowns c_i!*

Choose as $V_k = V_0 \oplus \bigoplus_{n=1}^{k} W_n$.

- Need multiresolution analysis defined on $[0, 1]$! Most constructions need to add *boundary functions* to avoid unwarrented boundary effects or need to *periodize* the problem.
- Due to its construction based on fractal functions, the DGHM multiwavelet is ideally suited for boundary value problems. Moreover, the DGHM multiwavelet can handle non-uniform geometries!
- Resulting matrix in linear system ill-conditioned.
- Preconditioning necessary; exact preconditioner known.
- Multiwavelet bases local bases (short support).
- Two parameters: scale and location.
- Ideal for detection of shocks and other singularities.
- Fast solvers for resulting preconditioned linear system.

6 Generalities

Biorthogonal Wavelets:

Instead of requiring

$$\langle \phi_{k\ell}, \phi_{k\ell'} \rangle = \delta_{\ell\ell'} I, \quad \langle \psi_{k\ell}, \psi_{k'\ell'} \rangle = \delta_{kk'} \delta_{\ell\ell'} I$$

ϕ and ψ are to satisfy the *biorthogonality conditions*

$$\boxed{\langle \phi_{k\ell}, \tilde{\phi}_{k\ell'} \rangle = \delta_{\ell\ell'} I, \quad \langle \psi_{k\ell}, \tilde{\psi}_{k'\ell'} \rangle = \delta_{kk'} \delta_{\ell\ell'} I} \tag{6.1}$$

with respect to the *dual bases* $\{\tilde{\phi}_{k\ell}\}$ and $\{\tilde{\psi}_{k\ell}\}$.

Advantages:

- More flexibility.
- Optimal convergence rates for certain integral equations.
- Construction of second generation wavelets.
- Shift regularity and approximation order back and forth between bases and dual bases.

Oblique Wavelets:

Projection onto subspaces not orthogonal but parallel to certain subspaces. (Cf. [1])

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